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# Fictitious crack model of concrete fracture

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*The substructure method introduced by Petersson is reformulated for the three-point bending specimen in order to obtain complete load-displacement relations without significant truncation. The problem of instability caused by the linearization of the softening in the fracture zone is discussed, and an alternative energy formulation is given so that it is possible to distinguish between stable and unstable situations. The reformulated substructure method is implemented on computer to give a multilinear stress crack opening displacement relation for the material in the fracture zone, and some qualitative results are given.*

## Introduction

The fracture mechanical properties of concrete in tension have been an important subject in concrete research over the last decade. The fracture properties have been studied both experimentally and theoretically by using fracture mechanical concepts.

Several models based on fracture mechanical ideas have been established to describe the fracture of concrete in tension. The fictitious crack model (FC model) formulated by Hillerborg and co-workers<sup>1-6</sup> is one of the most well known.

In the FC model a material point on the crack extension path is assumed to be in one of three possible states: an undisturbed elastic state (no fracture, no lack of compatibility), a fracture state in which the material is softened by microcracking (the fictitious crack), and a state of no stress transmission, where the point lies on a free surface.

The elastic state of all points in the body excluding those in the FC zone is described by the linear theory

of elasticity. The separation of points in the FC zone is described by a special constitutive relation, the so-called stress crack opening displacement relation ( $\sigma$ - $w$  relation) given by the function  $f(\cdot)$  defined in Fig. 1.

Petersson,<sup>4</sup> implemented the FC model on computer using the so-called substructure method. In effect, the body was partially cut through along the crack extension path. The problem was discretized by defining a finite number of nodes in which he satisfied the compatibility conditions in the elastic part of the body and the fracture conditions in the FC zone. The conditions were expressed by a set of linear equations. However, he did not treat the body as being divided into two separate substructures or sub-bodies, allowing the crack to extend to ultimate fracture. This leads to a significant truncation of the calculated force-displacement relation which is difficult to remove without introducing an unacceptably large number of nodes. The basic idea is sound, however, because the use of boundary nodes instead of a traditional finite-element method (FEM) technique speeds up the calculations tremendously. Based on the reduction of the number of nodes, the increase in speed can be estimated as approximately by a factor 100 for the problem considered. In the case of a larger body, the increase in speed will be even larger.

This Paper shows how the problem of truncation of the force-displacement relation can be overcome by effectively dividing the body into two sub-bodies using a displacement boundary value technique to solve the problem. This results in a fast and accurate algorithm well suited for simulation of tensile fracture problems. However, the method is limited to applications to problems where the crack path is known beforehand.

The Paper also gives an alternative method based on minimizing the total potential energy of the system, leading to a formulation which makes it possible to check the stability of the system in each incremental

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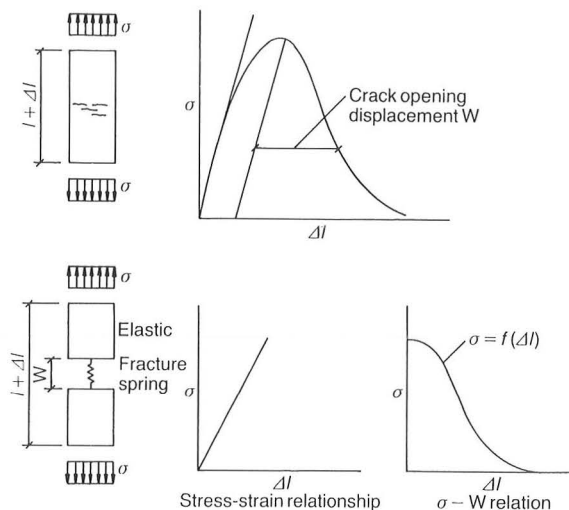


Fig. 1. Stress crack opening displacement relation for the material in the FC zone

step of the calculation of the force-displacement relation. Finally, the Paper describes how the reformulated substructure method is implemented on a computer in a way that makes it possible to use a multilinear  $\sigma$ - $w$  relation in the FC zone, and some qualitative results are given.

### Direct substructure method

A simply supported beam such as the standard test specimen proposed by RILEM<sup>7</sup> of length  $l$ , width  $d$  and height  $h$  loaded by a force  $F_1$  in the middle is considered (see Fig. 2(a)). A crack of length  $a$  is assumed to be present in the tensile side of the beam just beneath the applied load. When the load is applied to the beam, the crack extends, and a fracture zone of length  $c$  develops in front of the crack tip.

Instead of following Petersson,<sup>4</sup> who solved the problem like a contact problem by cutting the beam partially through at mid-section, the beam is divided into two separate sub-bodies or substructures  $A$  and  $B$ , as shown in Fig. 2(b). Instead of the stress condition given by  $F_1$ , a displacement condition  $\delta_1$  is applied at the middle, and a displacement condition  $\delta_2$  is applied at the right-hand support, transforming the original stress boundary conditions into a new set of boundary conditions given in terms of displacements.

For the body considered the stress solution will be symmetric, and advantage could be taken of this by considering only one of the substructures. However, in order to illustrate the applicability of the method to non-symmetric problems the whole body is considered. If the problem is non-symmetric, mixed-mode conditions in the fracture zone might occur, and the problem becomes substantially more complex. However, numerical investigations indicate that the crack chooses a path in such a way that the local symmetry is approximately preserved, i.e. the dominant mode is mode one.<sup>8</sup> Therefore it is not difficult to generalize the method to a more complex situation where the global symmetry is not preserved.

### Integral equations

Node co-ordinates and displacements are measured in the  $(x, y)$  and  $(u, v)$  co-ordinate systems, respectively (see Fig. 2(b)), the  $x$ -axis lying on the lower side of the beam. Assuming small displacements, the rigid-body displacements can be expressed as

$$u_0^A = 2 \frac{\delta_1}{l} y \quad (1)$$

$$u_0^B = \delta_2 - 2 \frac{\delta_1}{l} y \quad (2)$$

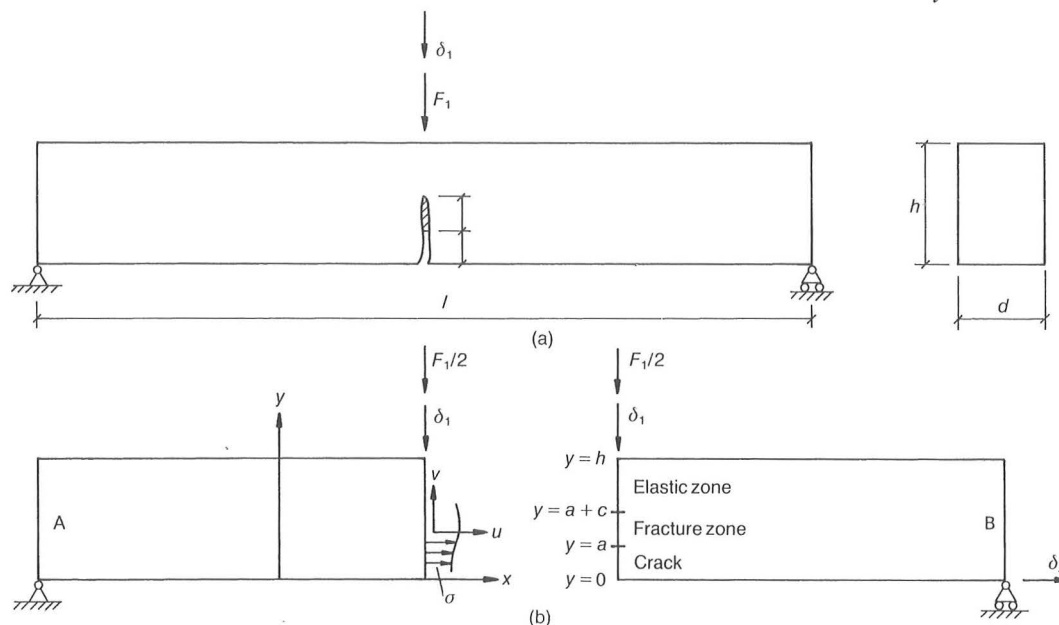


Fig. 2. Beam considered: (a) with crack of length  $a$  and fracture zone of length  $c$ ; (b) divided into substructures  $A$  and  $B$

where  $u_0^A$  and  $u_0^B$  are the displacements on the virtual surfaces created by dividing the original body into two new bodies. Let the stresses  $\sigma(y)$  be applied to the virtual surfaces. The displacements  $u_\sigma^A$  and  $u_\sigma^B$  caused by these stresses are given by

$$u_\sigma^A(y) = \int_a^h \sigma(y')g(y, y')dy' \quad 0 \leq y \leq h \quad (3)$$

$$u_\sigma^B(y) = -\int_a^h \sigma(y')g(y, y')dy' \quad 0 \leq y \leq h \quad (4)$$

where  $g(y, y')$  is Green's function for the displacements considered. The total displacement fields of the two bodies A and B are then given by

$$u^A = u_\sigma^A + u_0^A \quad (5)$$

$$u^B = u_\sigma^B + u_0^B \quad (6)$$

which yield the crack opening displacement

$$w(y) = u^B - u^A = -2 \int_a^h \sigma(y')g(y, y')dy' + \delta_2 - 4 \frac{\delta_1}{l} y \quad (7)$$

The conditions we have to satisfy are the compatibility condition

$$w(y) = 0 \Rightarrow$$

$$2 \int_a^h \sigma(y')g(y, y')dy' - \delta_2 + 4 \frac{\delta_1}{l} = 0$$

$$a + c \leq y \leq h \quad (8)$$

the fracture condition

$$f(w(y)) = \sigma(y) \Rightarrow$$

$$f\left(-2 \int_a^h \sigma(y')g(y, y')dy' + \delta_2 - 4 \frac{\delta_1}{l} y\right) = \sigma(y)$$

$$a \leq y \leq a + c \quad (9)$$

which determines the unknown stresses  $\sigma(y)$ , and finally the equilibrium condition

$$\int_a^h \sigma(y)dy = 0 \quad (10)$$

which determines the unknown displacement  $\delta_2$ .

### System of linear equations

The first step in establishing the system of linear equations is to discretize the integral equations, i.e. the co-ordinate  $y$  is restricted to the discrete values  $y_i$ ,  $i = 1, 2, \dots, n$ , and consequently the stresses  $\sigma(y)$  and Green's function  $g(y, y')$  are expressed in terms of the nodal forces  $s_i$  and Green's matrix  $g_{ij} = g(y_i, y_j)$ , respectively. The spacing  $a_0$  between nodes is assumed to be constant.

The conditions given in equations (8)–(10) can now be expressed as the sums

$$\sum_{j=k}^n g_{ij}s_j - \frac{\delta_2}{2} + 2 \frac{\delta_1}{l} y_i = 0 \quad m \leq i \leq n \quad (11)$$

$$f\left(-2 \sum_{j=k}^n g_{ij}s_j + \delta_2 - 4 \frac{\delta_1}{l} y_i\right) = s_i \quad k \leq i < m \quad (12)$$

$$\sum_{j=k}^n s_j = 0 \quad k \leq i \leq n \quad (13)$$

where  $k$  is the first node of the fracture zone,  $y_k = a$ , and  $m$  is the first node of the elastic zone,  $y_m = a + c$ . Taking  $f(\cdot)$  as a linear function

$$f(w) = f_0 + \alpha w \quad \alpha \leq 0 \quad (14)$$

equation (12) yields

$$f_0 - 2\alpha \sum_{j=k}^n g_{ij}s_j + \alpha\delta_2 - 4\alpha \frac{\delta_1}{l} y_i - s_i = 0$$

$$k \leq i < m \quad (15)$$

If  $f(\cdot)$  is not taken as a linear function, but as a piecewise linear function to approximate a more general  $\sigma$ - $w$  relation, the problem can still be expressed as a system of linear equations formed by equations (11), (13) and (15). In this case, different  $\alpha$ -values have to be used for the nodes according to where they are situated on the  $f$ -curve. The system of linear equations can then be written

$$Ax = b \quad (16)$$

where the coefficient matrix  $A$  is given by

$$A = \begin{bmatrix} 2\alpha_k g_{k,k} + 1 & 2\alpha_k g_{k,k+1} & \dots & 2\alpha_k g_{k,n} & -\alpha_k \\ 2\alpha_{k+1} g_{k+1,k} & 2\alpha_{k+1} g_{k+1,k+1} + 1 & \dots & 2\alpha_{k+1} g_{k+1,n} & -\alpha_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\alpha_{m-1} g_{m-1,k} & 2\alpha_{m-1} g_{m-1,k+1} & \dots & 2\alpha_{m-1} g_{m-1,n} & -\alpha_{m-1} \\ -g_{m,k} & -g_{m,k+1} & \dots & -g_{m,n} & 0.5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -g_{n,k} & -g_{n,k+1} & \dots & -g_{n,n} & 0.5 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \quad (17)$$

and where  $x$  and  $b$  are given by

$$x = \begin{bmatrix} s_k \\ s_{k+1} \\ \vdots \\ s_{m-1} \\ s_m \\ \vdots \\ s_n \\ \delta_2 \end{bmatrix} \quad (18)$$

$$b = \begin{bmatrix} f_{0,k} - 4\alpha_k \frac{\delta_1}{l} y_k \\ f_{0,k+1} - 4\alpha_{k+1} \frac{\delta_1}{l} y_{k+1} \\ \vdots \\ f_{0,m-1} - 4\alpha_{m-1} \frac{\delta_1}{l} y_{m-1} \\ 2 \frac{\delta_1}{l} y_m \\ \vdots \\ 2 \frac{\delta_1}{l} y_n \\ 0 \end{bmatrix} \quad (19)$$

The non-linearity of the problem is then introduced by updating the coefficient matrix  $A$  and the right-hand side  $b$  according to the movements of the nodes on the  $f$ -curve. The problems of keeping track of where the nodes are situated and the problems of updating  $A$  and  $b$  are treated in the section on implementation. It must be noted, however, that equation (14) expresses the crack opening displacement relations for the nodal forces and not for the stresses. This means that the constitutive parameters  $f_0$  and  $\alpha$  for the first node  $i = 1$  and the last node  $i = n$  have to be multiplied by a factor 0.5 to correct for the smaller areas corresponding to these nodes.

When the nodal forces are determined by solving the system of linear equations described above, the total force  $F_1$  is obtained from the equilibrium condition

$$\frac{1}{4} F_1 l + \sum_{j=k}^n s_j y_j = 0 \quad (20)$$

and the crack opening displacements  $w_i$  are given by

$$w_i = -2 \sum_{j=k}^n g_{ij} s_j + \delta_2 - 4 \frac{\delta_1}{l} y_i \quad k \leq i < m \quad (21)$$

### Energy substructure method

In the preceding section, a system of linear equations

was established by forming a coefficient matrix  $A$  containing essentially Green's coefficients  $g_{ij}$  and the discrete spring constants  $\alpha_i$  describing the local properties of the crack opening displacement relation. In this case, where the spring stiffnesses  $\alpha$  are less than zero the total potential energy might become non-positive definite; i.e. the solutions found by solving system equation (16) do not correspond to a true minimum for the potential energy. In this case no stable solution exists, and the simulation procedure becomes unstable. Experience shows that this is a serious problem. The problem is due to the piecewise linear approximation to the softening problem in the fracture zone, and reflects the poor approximation of the Hillerborg/Petersson model in the case of unloading.

The standard way to check for system stability is to require that  $x^T A x \geq 0$  for all  $x$ ; i.e. the matrix  $A$  has to be positive definite. However, this is only meaningful if  $x^T A x$  can be interpreted as the quadratic part of the total potential energy of the system. In this case, where the sequence of the equations (rows can be interchanged) and the sign of the coefficients of a given row are arbitrary, this is clearly not so. Using the formulation given above, there is no simple way to check that the energy is positive definite, and therefore no simple way to investigate when and why the simulation procedure becomes unstable.

If a safe way to check for positive definiteness of the energy is needed, another formulation has to be given. One method is to express the total potential energy of the system and then obtain the solution by requiring that the potential energy is minimized. In this case, a system of linear equations is obtained for which it is only required that the corresponding system matrix is positive definite. Here the whole beam is considered, but advantage is taken of the symmetry, and therefore only part A of the original beam is considered, as shown in Fig. 3.

The total potential energy can be written as

$$U = U_F + U_c + U_s \quad (22)$$

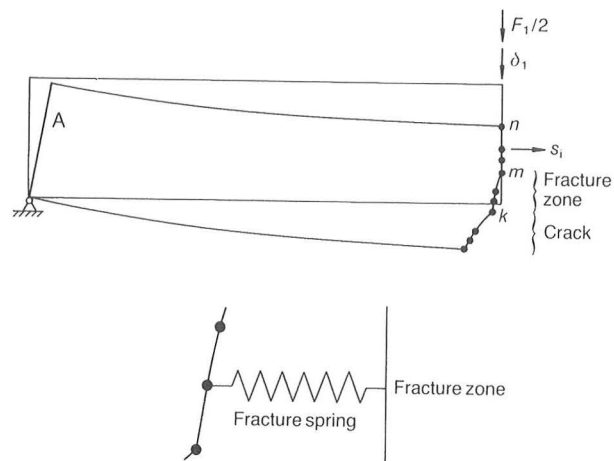


Fig. 3. Beam considered for the energy analysis

where  $U_F$  is the potential of the load  $F_1$ ,  $U_e$  is the strain energy of the body and  $U_s$  is the strain energy of the springs describing the crack opening in the fracture zone. Using the equilibrium equation (20) the potential of the load  $F_1$  is given by

$$U_F = -\frac{1}{2}F_1\delta_1$$

$$= 2\frac{\delta_1}{l}\sum_{j=k}^n s_j y_j \quad (23)$$

The strain energy of the body is

$$U_e = \frac{1}{2}\sum_{i=k}^n s_i \sum_{j=k}^n g_{ij}s_j \quad (24)$$

The strain energy of the fracture springs is slightly more difficult to determine. Equation (14) can be written as

$$s_i = \alpha_i(w_i - w_{0i}) \quad w_{0i} = -\frac{f_{0i}}{\alpha_i} \quad (25)$$

which yields the strain energy

$$U_s = \frac{1}{2}\sum_{j=k}^{m-1} \alpha_j(w_j - w_{0j})^2 \quad (26)$$

Now substituting the expressions for  $w_j$  and  $w_{0j}$  into equation (26) yields

$$U_s = \frac{1}{2}\sum_{i=k}^{m-1} \alpha_i \left( \sum_{j=k}^n g_{ij}s_j + \frac{f_{0i}}{\alpha_i} \right)^2 \quad (27)$$

and the total potential energy then becomes

$$U = 2\frac{\delta_1}{l}\sum_{i=k}^n s_i y_i + \frac{1}{2}\sum_{i=k}^n s_i \sum_{j=k}^n g_{ij}s_j$$

$$+ \frac{1}{2}\sum_{i=k}^{m-1} \alpha_i \left( \sum_{j=k}^n g_{ij}s_j + \frac{f_{0i}}{\alpha_i} \right)^2 \quad (28)$$

The system of linear equations is now obtained by requiring that  $\partial U/\partial s_q = 0$  for  $k \leq q \leq n$

$$2\frac{\delta_1}{l}y_q + \sum_{j=k}^n g_{qj}s_j + \sum_{i=k}^{m-1} \alpha_i \left( \sum_{j=k}^n g_{ij}s_j + \frac{f_{0i}}{\alpha_i} \right) g_{iq} = 0$$

$$k \leq q < n \quad (29)$$

which yields the matrix equation

$$Cs = d \quad (30)$$

where the vector  $\bar{s}$  contains the nodal forces  $s_i$ , and where the matrix elements  $c_{ij}$  and the elements of the right-hand side  $d_i$  are given by

$$c_{ij} = g_{i+k,j+k} + \sum_{p=k}^{m-1} g_{i+k,p} \alpha_p g_{p,j+k}$$

$$0 \leq i, j < n - k \quad (31)$$

and

$$d_i = -2\frac{\delta_1}{l}y_i - \sum_{j=k}^{m-1} g_{ij}f_{0j} \quad 0 \leq i < n - k \quad (32)$$

Here the matrix elements  $c_{ij}$  and the elements of the right-hand side  $d_i$  are numbered  $c_{00}, c_{01}, \dots$  and  $d_0, d_1, \dots$ , respectively. It is easy to see that the compliance matrix  $C$  is symmetric and that  $s^T C s$  represents the quadratic part of the potential energy. Therefore, using the energy substructure method one only has to show that the compliance matrix  $C$  is positive definite to be sure that the system is stable.

It should be noted that the fracture spring stiffness coefficients  $\alpha_i$  must be multiplied by an additional factor 2, since the length of the fracture springs in this formulation is only half the real length.

## Implementation

Only the direct substructure method has been implemented, but it might as well have been the energy substructure method.

In the following the incremental step used for calculation of a typical point on the load deflexion relation for the considered beam is explained. The nodes in the FC zone are all lying on the  $\sigma$ - $w$  relation as shown in Fig. 4, denoting the nodal forces and nodal crack opening displacements by  $s_j$  and  $w_j$ , respectively,  $k \leq j < m$ .

The parameter controlling the problem is, as mentioned earlier, given by the beam deflexion  $\delta_1$ . If the beam deflexion is increased, all points lying on the  $\sigma$ - $w$  relation will be moving to the right, i.e. towards larger  $w_j$  values. The problem is to determine which point is the first to reach a kink on the  $\sigma$ - $w$  relation, because when a point is crossing a kink point on the multi-linear  $\sigma$ - $w$  relation the system matrix and the right-hand side of the system of linear equations, see equations (17)–(19), have to be updated, and the system is said to change the state of fracture.

Each of the points lying on the fracture relation have a nearest kink point when the point is moving to the right on the curve. The crack opening displacements of these nearest kink points are denoted  $w_j^*$ , see Fig. 4. In order to determine which point is the first

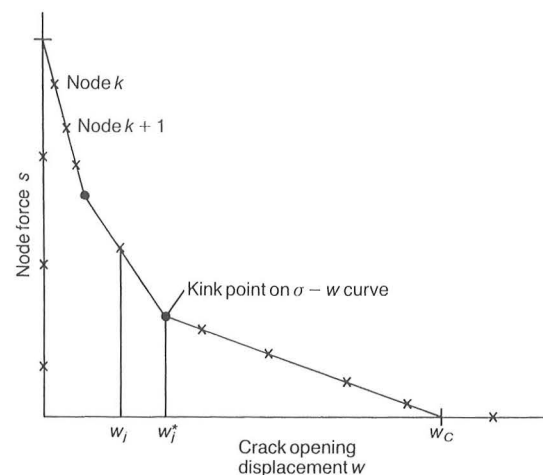


Fig. 4. Location of the nodes on the stress crack opening relation

to cross a kink on the fracture relation, the beam deflexion is given a small increment  $d\delta_1$ , and the corresponding increments  $dw_j$  for all the nodes lying on the fracture relation are determined by solving the system of linear equations using the system matrix and the right-hand side corresponding to the present state of fracture. Then the sensitivities

$$\mu_j = \frac{dw_j}{w_j^* - w_j} \quad (33)$$

can be calculated. For the first elastic node,  $j = k$ , the sensitivity is calculated in a similar way using nodal forces instead of crack opening displacements

$$\mu_k = \frac{ds_k}{s_k^* - s_k} \quad (34)$$

Here the kink point corresponds to the beginning of the fracture relation, and  $s_k^*$  is therefore equal to the tensile strength of the nodes. The point  $j^*$  having the largest sensitivity  $\mu^*$  is then moved to the nearest kink point by solving the system of linear equations using the beam deflexion increment

$$\Delta\delta_1^* = \frac{d\delta_1}{\mu^*} \quad (35)$$

and using the system matrix and the right-hand side for the present state of fracture. Then the system matrix and the right-hand side are updated, and the next point on the force displacement curve can be calculated. If the crack opening displacement for a node exceeds the ultimate crack width  $w_c$ , the node is removed from the set of nodes in the fracture zone; i.e. there is a real crack and no stress can be transmitted.

This procedure continues until there is only one elastic node left and that node is the next one to be moved into the fracture zone. The algorithm outlined above was implemented on a personal computer, and some qualitative relations were investigated.

Table 1. Geometry and material properties for the beams analysed

Height: mm	80
Width: mm	40
Length: mm	400
Elasticity modulus $E$ : N/mm <sup>2</sup>	32 550
Fracture energy $G_f$ : N/m	109.6
Tensile strength $f_t$ : N/mm <sup>2</sup>	2.86

## Results

Two problems were investigated: the problem of sensitivity to the number of nodes across the beam section, and the problem of sensitivity to the degree of approximation for the stress crack displacement opening relation.

The sensitivity problems were analysed using small-size beams. The geometry of the beams and the material properties are listed in Table 1.

The material properties are taken as the average properties for the concretes tested by Wolinski *et al.*<sup>9</sup> The stress crack opening relation measured by Wolinski *et al.* is shown in Fig. 5 together with the approximations used in the sensitivity analysis.

Analysis of small beams was chosen because of the problems of stability discussed earlier. Owing to the steep fracture relation shown in Fig. 5 it was necessary to use small beams in order to ensure that all the cases could be analyzed using the same beam size. It is not difficult to see that the problem of stability increases with the size of the beam.

It is clear that the steeper the fracture relation the larger the risk of instability. The dependence on the size of the beam can be obtained by simple dimensional analysis. The physical quantities influencing the stability of the beam are assumed to be the size  $l$ , the tensile strength  $f_t$ , the elastic properties given by  $g_{ij}$  and the fracture properties of the material in the fracture zone described by  $f_0$  and  $\alpha$ . Accepting these assumptions, it

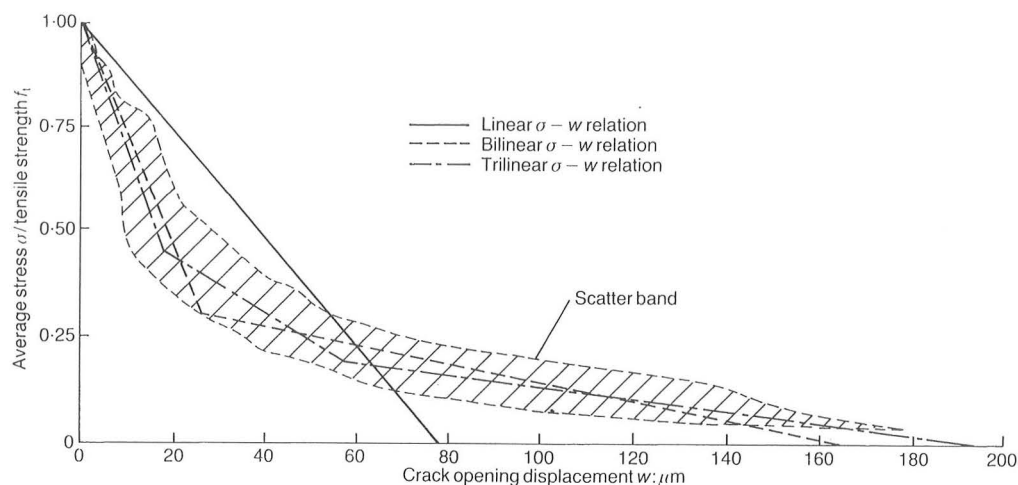


Fig. 5. Stress crack opening relation, taken from reference 9



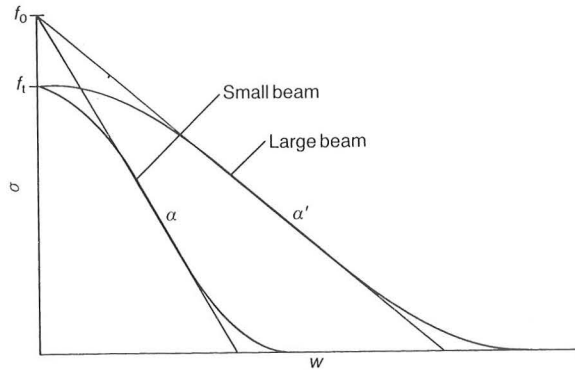


Fig. 6. Fracture model law

is easy to see that the problem is described by three dimensionless products, and that these can be chosen as

$$\pi_1 = g_{ij} f_t l \quad (36)$$

$$\pi_2 = \frac{\alpha l}{f_t} \quad (37)$$

$$\pi_3 = \frac{f_0}{f_t} \quad (38)$$

Now let us consider another beam with the corresponding quantities  $l', f_t', g_{ij}', \alpha', f_0'$ . From the model invariance of the  $\pi$ -products it is seen that the

similitude requirements for the beam in the case of the same tensile strength,  $f_t' = f_t$ , are given by

$$f_0' = f_0 \quad (39)$$

$$g_{ij}' = g_{ij} \frac{l}{l'} \quad (40)$$

$$\alpha' = \alpha \frac{l}{l'} \quad (41)$$

From these results it can be seen that if  $l' > l$  then the slope of the fracture curve  $\alpha'$  must decrease; i.e. the material has to be tougher in order to ensure the same behaviour of the two beams considered, see Fig. 6. This means that if the fracture properties of the material are kept constant and the beam size is decreased then the risk of instability becomes smaller.

In the investigation of sensitivity to the number of nodes the cross-section was divided into 5, 10, 15 and 19 nodes, and the influence coefficients  $g_{ij}$  were determined by linear finite-element analysis using constant strain elements. The load displacement relations were calculated using a linear stress crack opening displacement relation, and the results for the four cases are shown in Fig. 7. It can be seen that only the curve for the coarse mesh containing only five nodes differs significantly from the others, indicating that the

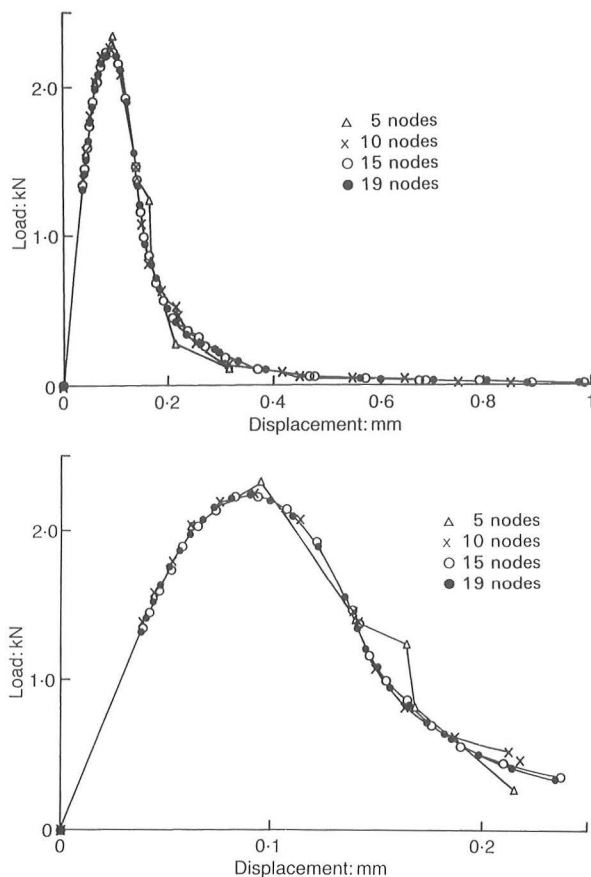


Fig. 7. Sensitivity to the number of nodes

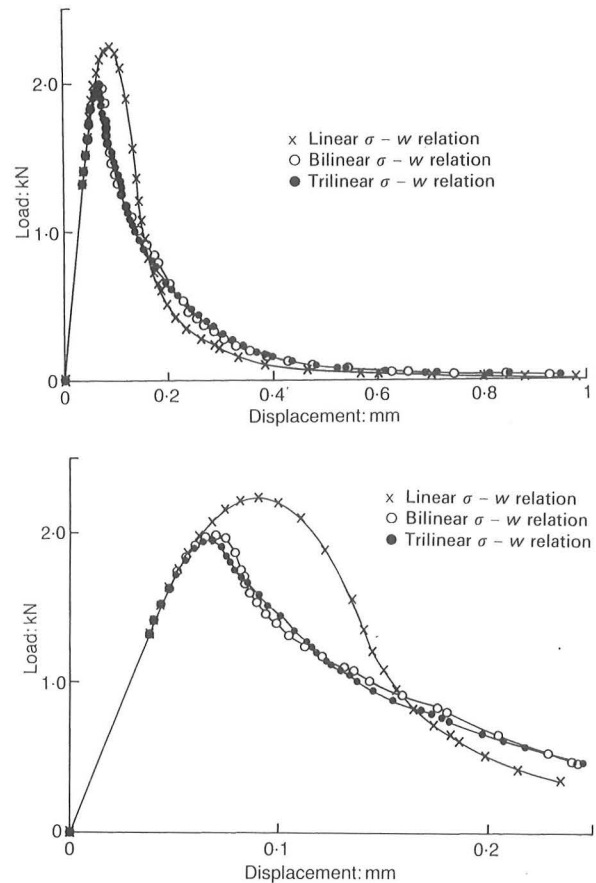


Fig. 8. Sensitivity to the degree of approximation of the stress crack opening displacement relation



method is not very sensitive to the chosen number of nodes.

Three different approximations have been used to evaluate the sensitivity of the method to the degree of approximation of the stress crack opening displacement relation. The measured  $\sigma$ - $w$  relation and the three different approximations are shown in Fig. 5. The results from the sensitivity analysis are shown in Fig. 8.

It is evident that the shape of the stress crack opening displacement relation has significant influence on the results. Improving the approximation from a linear (no kinks) to a bilinear (one kink) stress crack opening displacement relation causes a drop in the calculated ultimated load of approximately 10%, and the shape of the load displacement relation is changed significantly. The approximation by a trilinear (two kinks) fracture relation, however, does not seem to change the results significantly, indicating the sufficiency of the bilinear approximation.

## Conclusions

On the basis of the experience with the reformulated substructure method the following conclusions can be drawn.

1. The method is able to simulate crack growth far beyond the limits of the known substructure method, revealing results without truncations of significance on the load-displacement relation.

2. The results are not very dependent on the number of nodes in the crack extension path, and relatively rough discretizing can be used.

3. It is important for the results that the shape of the stress crack opening relation is modelled approximately correctly. However, it is not necessary to use a multilinear relation. A bilinear relation seems to be sufficient.

4. A serious problem using the substructure method is that the system becomes unstable too easily owing to the simple local linearization of the stress crack opening displacement relation. The problem is partly solved by using the proposed energy formulation, which allows for an easy check on when and

why the simulation becomes unstable, but it is expected that the problem can be removed only by a better modelling of unloading of the material in the fracture zone.

5. The energy formulation should be used in future applications because of its simple way of checking for instability, and because of its symmetric compliance matrix.

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